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Classification of configurations of systems of identical particles on finite sets

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Abstract. All possible configurations of a system of N identical particles are classified for the case when the configuration space Q of a single particle reduces to a finite set. The classification is achieved by means of a combinatorial analogue of the duality of Weyl. Two groups: the Pauli group permuting identical particles, and the symmetric group on Q, act on the Nth Cartesian power of Q. These actions mutually commute, and their orbit structure is classified by appropriate epikernels. Moreover, each stratum of each group is invariant under the dual action, which yields a coarsening of permutation representations, acting on appropriate sets of orbits. It is shown that such a coarsening recovers the geometric symmetry for the case when particles are indistinguishable.

1. Introduction

The duality of Weyl (1950) has been described as the framework for a quantum theory of identical particles. The key ingredient of this framework is commutativity of actions of two groups, the symmetric group $\sum_N$ on the set of N identical particles (referred to hereafter as the Pauli group), and the unitary group $U(n)$—the quantum symmetry group of an n-dimensional space L of quantum states of a single particle. Both groups, $\sum_N$ and $U(n)$, act on the Nth tensor power space $L^{\otimes N}$, and the commutativity of actions of these two groups yields a classification of vectors of an orthonormal basis in this space in terms of irreducible representations of both groups as a compatible and complete set of quantum numbers. This framework has been succesfully applied, by use of Wigner–Racah calculus (Wigner 1959, Biedenharn and Louck 1981), to the theory of spectra of multielectron atoms (Wybourne 1965, Jucys and Savukynas 1973) and nuclear shells (Vanagas 1971). Moshinsky and Quesne (1970, 1971; cf also Quesne 1973, 1975, Aguilera-Navarro and Aguilera-Navarro 1975) extended the duality of Weyl in terms of the notion of complementary groups, defined within a single irreducible representation [1] of the 'supergroup' $U(2^n)$, the quantum symmetry group of the 2^n-dimensional Fock space of fermions with the single-particle space L, $\dim L = n$, and $0 \leqslant N \leqslant n$, or the corresponding irreducible representation $[(\frac{1}{2})^n]$ of the 'noninvariance group' $O^+(2N+1) \subset U(2^N)$, generated by all creation and anihilation operators. There are also several attempts for use of such an approach in the theory of molecules (Florek *et al* 1991, Paldus 1976, Kim 1981, Newman 1981, Michel and Mozrzymas 1982, Butler *et al* 1983).

A somehow distinct approach to the theory of identical particles is taken within the concept of anyons and intermediate statistics (Leinaas 1993, Einarsson 1995). Anyons are hypothetical particles living on two-dimensional manifolds Q, and their indistinguishability

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is accounted for by the braid group on the manifold rather than the Pauli group $\sum_N$. The classical configuration space of a system of N anyons is constructed as the space of regular orbits of the Pauli group $\sum_N$ on the Nth Cartesian power $Q^{\times N}$ of the single-particle manifold Q, and the non-trivial homotopy of this orbit space is the source of an intermediate statistics.

In this paper we propose a version of the duality of Weyl for the case when the configuration space Q is a finite set. Our version is based on actions of finite groups on finite sets and is purely combinatorial, in particular it does not appeal to any linear or unitary structure. We aim to point out that this scheme also provides some interesting clasification of symmetry types of points of N-particle configuration spaces and, in particular, it reproduces the geometric symmetry of some manifolds of N-particle states, as a result of coarsening of the group actions to appropriate orbit spaces.

The combinatorial version of the Weyl duality presented here is also well adapted to the theory of identical particles on lattices for both approaches: the Wigner–Racah approach to multielectron multicentre systems, and to anyons.

2. The combinatorial version of the duality of Weyl

Let Q be a finite set of $|Q| = n$ elements, equipped with the interpretation of the configuration space of a single particle. Let $H \subset \sum_n$ be a subgroup of the symmetric group $\sum_n$ on the set Q, e.g. H might be the geometric symmetry group of Q. We consider the system of N identical itinerant particles with positions $q \in Q$, allowed to hop between some different points q and q' in Q. The natural candidate for the set of all N-particle positions is the Nth Cartesian power set

$$Q^{\times N} = \{(q_1, \dots, q_N) | q_j \in Q,\ j \in \tilde{N}\} \tag{1}$$

where

$$\tilde{N} = \{j = 1, 2, \dots, N\} \tag{2}$$

is the set of labels of particles. Let $G \subset \sum_N$ be subgroup of the Pauli group $\sum_N$—the symmetric group on the set $\tilde{N}$.

We now introduce two actions, $A : G \times Q^{\times N} \to Q^{\times N}$, and $F : H \times Q^{\times N} \to Q^{\times N}$, of the group G and H respectively, on the set $Q^{\times N}$, specified by

$$A(g) = \begin{pmatrix} f \\ f \circ g^{-1} \end{pmatrix} \qquad g \in G \qquad f \in Q^{\times N} \tag{3}$$

and

$$F(h) = \begin{pmatrix} f \\ h \circ f \end{pmatrix} \qquad h \in H \qquad f \in Q^{\times N} \tag{4}$$

where group elements $g \in G \subset \sum_N$ and $h \in H \subset \sum_N$ are considered as mappings $g : \tilde{N} \to \tilde{N}$ and $h : Q \to Q$, respectively, $f \in Q^{\times N}$ is the mapping $f : \tilde{N} \to Q$ defined by $f(j) = q_j$, $j \in \tilde{N}$, and $f \circ g^{-1}$, $h \circ f$ are compositions of mappings. The upper symbol in the parentheses in the right-hand side of equations (3) and (4) denotes an object of the permutation from the corresponding left-hand side, whereas the lower symbol is its image. In other words, we consider actions A and F, of two different groups, G and H, respectively, on the set $Q^{\times N}$ of all functions on $\tilde{N}$, valued in the single-particle configuration space Q. These two actions commute, i.e.

$$A(g)F(h) = F(h)A(g) = \begin{pmatrix} f \\ h \circ f \circ g^{-1} \end{pmatrix} \qquad h \in H \qquad g \in G \qquad f \in Q^{\times N}. \tag{5}$$

We recall here that the original scheme of Weyl duality corresponds to replacing the finite set Q by a finite-dimensional unitary space L of quantum states of a single particle, with n = dim L, and to putting $G = \sum_N$ and $H = U(n)$, with simultaneous substitution of $Q^{\times N}$ by $L^{\otimes N}$. Then the commutativity (5) yields a complete classification of basis states in the Nth tensor power space $L^{\otimes N}$ by irreducible representations of $\sum_N$ and $U(n)$ (Young diagrams), and appropriate degeneracy labels (Young and Weyl standard tableaux (Kerber 1991)). An important observation is that the linear space of all subspaces in $L^{\otimes N}$, transforming in accordance with a single irreducible representation of one of the groups, $\sum_N$ or $U(n)$, is invariant under the action of the other group. We point out in the next section the combinatorial analogue of this observation.

3. Stratification of a set under two commutative actions

In this section we analyse the structure, imposed on the set $Q^{\times N}$ by actions A and F, and resulting from the commutativity (5) of these actions.

We recall to begin with some definitions and facts, related to 'actions of groups on sets' (Kerber 1991, Michel 1980). Let $P : K \times \Omega \to \Omega$ denote the action of a group K on the set Ω. An *orbit* generated by P from an element $\omega \in \Omega$ is the set $O_K[\omega] = \{P(k)\omega | k \in K\} \subset \Omega$. The *stabilizer* of an element $\omega \in \Omega$ is a subgroup $K^\omega = \{k \in K | P(k)\omega = \omega\} \subset G$.

The restriction $P|_{O_K[\omega]}$ of the action P of the group K to an orbit $O_K[\omega]$ is called a transitive representation, and denoted $R^{K:K^\omega}$. Each stabilizer $K^\omega, \omega \in \Omega$, is an element of the lattice $\tilde{L}(K)$ of all subgroups of K. By a *lattice* we mean here a partially ordered set with the unique maximal and minimal element; the partial order in $\tilde{L}(K)$ is defined by inclusion $K^\omega \subset K$, and the unique elements are $\{e\}$ and K as the minimal and maximal, respectively. When ω runs over an orbit $O[\omega_o]$, then the stabilizer K^ω runs over the set

$$\kappa = [K^{\omega_o}] = \{kK^{\omega_o}k^{-1} | k \in K\} \subset \tilde{L}(K) \tag{6}$$

of all those subgroups of K, which are conjugated to K^{ω_o} in K. The set κ is referred to as the *epikernel* of the orbit $O[\omega_o]$. Each epikernel κ is an element of the lattice $\tilde{l}(K)$ of all classes of conjugated subgroups; the partial order in $\tilde{l}(K)$ is inherited from $\tilde{L}(K)$. The set of all orbits with the epikernel κ is called the *stratum* of the action P, corresponding to κ, The stratification of the set Ω under the action P of the group K is written in the form

$$\Omega/K = \bigcup_{\kappa \in ekP} S(P, \kappa) \tag{7}$$

where Ω/K denotes the set of all orbits of K on Ω, and $ekP \subset \tilde{l}(K)$ is the subset of all those epikernels of K which correspond to non-empty strata $(S(P, \kappa) \neq \emptyset\}$. We denote by

$$\bar{S}(P, \kappa) \bigcup_{\delta \in S(P,\kappa)} \delta \subset \Omega \tag{8}$$

the union of all orbits in the stratum $S(P, \kappa)$, so that

$$\Omega = \bigcup_{\kappa \in ekP} \bar{S}(P, \kappa) \tag{9}$$

is the decomposition of the set Ω into non-empty and pairwise disjoint subsets corresponding to different epikernels.

We formulate the combinatorial analogue of the duality of Weyl in terms of notions defined above by putting $\Omega = Q^{\times N}$ and taking by P either of the mutually dual actions A or F. Commutativity of these actions implies that each union

$$\bar{S}(A,\gamma) = \bigcup_{\alpha\in S(A,\gamma)} \alpha \subset Q^{\times N} \tag{10}$$

of all orbits of G on $Q^{\times N}$ with the epikernel $\gamma \in ekA$ is an invariant subset of the action F. Dually, each union

$$\bar{S}(F,\eta) = \bigcup_{\beta\in S(F,\eta)} \beta \subset Q^{\times N} \tag{11}$$

of all orbits of H on $Q^{\times N}$ with the epikernel $\eta \in ek\,F$ is an invariant subset of the action A.

We have therefore

$$\bar{S}(A,\gamma)/H = \bigcup_{\eta\in ek\,F} T(A\gamma,\eta) \qquad \gamma \in ekA \tag{12}$$

and

$$\bar{S}(F,\eta)/G = \bigcup_{\gamma\in ekA} T(F\eta,\gamma) \qquad \eta \in ek\,F \tag{13}$$

where $T(A\gamma,\eta)$ is the stratum of the restriction $F|_\gamma$ of the action F to the invariant subset $\bar{S}(A,\gamma) \subset Q^{\times N}$, and dually, $T(F\eta,\gamma)$ is the stratum of the restriction $A|\eta$ of the action A to the invariant subset $\bar{S}(F,\eta) \subset Q^{\times N}$. Clearly, the corresponding unions of orbits coincide, i.e.

$$\bar{T}(A,\gamma,\eta) = \bigcup_{\beta\in T(A,\gamma,\eta)} \beta = \bigcup_{\alpha\in T(F\eta,\gamma)} \alpha = \bar{T}(F\eta,\gamma). \tag{14}$$

We thus put

$$\bar{T}(\gamma,\eta) = \bar{T}(A\gamma,\eta) = \bar{T}(F\eta,\gamma) \subset Q^{\times N} \tag{15}$$

and refer to $\bar{T}(\gamma,\eta)$ as to the *region* in $Q^{\times N}$, corresponding to the pair (γ,η) of epikernels of dual actions.

Stratification of the Nth Cartesian power $Q^{\times N}$ under the two dual actions A and F results therefore in the decomposition

$$Q^{\times N} = \bigcup_{\substack{\gamma\in ek\,A\\ \eta\in ek\,F}} \bar{T}(\gamma,\eta) \tag{16}$$

into pairwise disjoint regions, characterized by epikernels of both actions.

4. Coarsened actions

Actions A and F on the set $Q^{\times N}$ generate in a natural way new actions, $A' : G\times Q^{\times N}/H \to Q^{\times N}/H$ and $F' : H \times Q^{\times N}/G \to Q^{\times N}/G$, respectively, on the corresponding set of orbits of the dual group. Thus we have

$$A'(g) = \begin{pmatrix} \beta[f] \\ \beta[A(g)f] \end{pmatrix} \qquad f \in \beta \subset Q^{\times N} \qquad g \in G \qquad (\beta \in Q^{\times N}/H) \tag{17}$$

and

$$F'(h) = \begin{pmatrix} \alpha[f] \\ \alpha[F(h)f] \end{pmatrix} \qquad f \in \alpha \subset Q^{\times N} \qquad h \in H \qquad (\alpha \in Q^{\times N}/G). \tag{18}$$

One can easily prove that formulae (17) and (18) do not depend on the choice of an orbit representative $f \in \alpha$, or $f \in \beta$, and that they define the actions A' of G, and F' of H, on the set of orbits of the dual action. We refer hereafter to A' and F' as to the *coarsened* actions.

Clearly, each set $T(A\gamma, \eta) \subset Q^{\times N}/H$ is invariant under the coarsened action A', and dually, each set $T(F\eta, \gamma) \subset Q^{\times N}/G$ is invariant under F'. Thus the whole structure of the coarsened actions A' and F' can be determined at the level of each region $\bar{T}(\gamma, \eta)$ separately. The elementary building brick is a *block* $b \subset \bar{T}(\gamma, \eta)$, defined as an orbit of the direct product group $G \times H$. Thus the block $b[f]$, generated from an element $f \in \bar{T}(\gamma, \eta) \subset Q^{\times N}$, is defined by

$$b[f] = \{A(g)F(h)f \mid g \in G \qquad h \in H\} \subset \bar{T}(\gamma, \eta). \tag{19}$$

In addition to the pair (γ, η) which characterizes the region $\bar{T}(\gamma, \eta)$, each block b is characterized by another pair (γ', η') of epikernels $\gamma' \in \tilde{l}(G), \eta' \in \tilde{l}(H)$. They are defined as follows. Let $f \in b \subset \bar{T}(\gamma, \eta)$ and let

$$G^{H,f} = \{g \in G \mid A(g)f \in O_H[f]\} \tag{20}$$

$$H^{G,f} = \{h \in H \mid F(h)f \in O_G[f]\} \tag{21}$$

be subgroups of G and H, respectively, consisting of all those elements which carry f into an element belonging to the same orbit of the dual group as f. Then the new epikernels defined by

$$\gamma' = [G^{H,f}] \qquad \eta' = [H^{G,f}] \tag{22}$$

do not depend upon the choice of f within the block b. The subgroups (20) and (21) also determine chains

$$G^f \subseteq G^{H,f} \subseteq G \tag{23}$$

and

$$H^f \subseteq H^{G,f} \subseteq H \tag{24}$$

with the property

$$\frac{|G^{H,f}|}{|G^f|} = \frac{|H^{G,f}|}{|H^f|} = c(\gamma\gamma', \eta\eta'). \tag{25}$$

The positive integer $c(\gamma\gamma', \eta\eta')$ characterizes the block b and will be referred to hereafter as the degree of coarsening on the block b. The integers

$$m_H(\gamma\gamma', \eta\eta') = \frac{|G|}{|G^{H,f}|} \tag{26}$$

and

$$m_G(\gamma\gamma', \eta\eta') = \frac{|H|}{|H^{G,f}|} \tag{27}$$

are multiplicities of the transitive representation $R^{H:H^f}$ and $R^{G:G^f}$ respectively on the block b. The total number of elements of the block b is

$$|b| = \frac{|G| \cdot |H|}{|H^{G,f}| \cdot |G^{H,f}|} \cdot c(\gamma\gamma', \eta\eta') = \frac{|G| \cdot |H|}{|G^f| \cdot |H^f| \cdot c(\gamma\gamma', \eta\eta')}. \tag{28}$$

We can thus write down restrictions of both actions, A and F, to the block b, as

$$A|_b \cong \frac{|H|}{|H^{G,f}|} R^{G:G^f} \qquad G^f \in \gamma \qquad H^{G,f} \in \eta' \tag{29}$$

and

$$F|_b \cong \frac{|G|}{|G^{H,f}|} R^{H:H^f} \qquad H^f \in \eta \qquad G^{H,f} \in \gamma'. \tag{30}$$

The structure (29)–(30) of actions A and F on the block b readily implies

$$A' \,|_{b/H} \cong R^{G:G^{H,f}} \qquad G^{H,f} \in \gamma' \in \tilde{l}(G) \tag{31}$$

and

$$F' \,|_{b/G} \cong R^{H:H^{G,f}} \qquad H^{G,f} \in \eta' \in \tilde{l}(H). \tag{32}$$

Thus coarsened actions are transitive on each block b, and their stabilizers ($G^{h,f}$ or $H^{G,f}$) increase in comparison to stabilizers of initial actions (resp. G^f or H^f), in accordance with the degree $c(\gamma\gamma', \eta\eta')$ of coarsening.

The above construction suggests that we divide each region $\bar{T}(\gamma, \eta)$ into subregions $\bar{D}(\gamma\gamma', \eta\eta')$, each subregion consisting, by definition, of all blocks characterized by the tetraiad $(\gamma\gamma', \eta\eta')$ of epikernels. Then the structure imposed on the set $Q^{\times N}$ by actions A and F can be summarized in the decomposition

$$Q^{\times N} = \cup_{\gamma,\gamma' \in \tilde{l}(G)} \cup_{\eta,\eta' \in \tilde{l}(H)} \bar{D}(\gamma\gamma', \eta\eta'). \tag{33}$$

Restriction of summation in (33) to γ', η' yields regions $\bar{T}(\gamma, \eta)$, whereas the sum over $\gamma, \gamma'\eta'(\eta, \gamma', \eta')$ yields the union $\bar{S}(F, \eta)(or\, \bar{S}(A, \gamma))$ of orbits in the stratum $S(F, \eta)$ (or $S(A, \gamma)$).

Let $\mu(\gamma\gamma', \eta\eta')$ be the number of blocks in the subregion $\bar{D}(\gamma\gamma', \eta\eta')$. Then the decomposition of initial actions into transitive representations reads

$$A \cong \sum_{\gamma,\gamma' \in \tilde{l}(G)} \sum_{\eta,\eta' \in \tilde{l}(u)} \mu(\gamma\gamma', \eta\eta') \frac{|H|}{|H^{G,f}|} R^{G:G^f} \tag{34}$$

and

$$F \cong \sum_{\gamma,\gamma' \in \tilde{l}(G)} \sum_{\eta,\eta' \in \tilde{l}(H)} \mu(\gamma\gamma', \eta\eta') \frac{|G|}{|G^{H,f}|} R^{H:H^f} \tag{35}$$

in accordance with the decomposition (33), whereas the coarsened actions have the form

$$A' \cong \sum_{\gamma,\gamma' \in \tilde{l}(G)} \sum_{\eta,\eta' \in \tilde{l}(H)} \mu(\gamma\gamma', \eta\eta') R^{G:G^{H,f}} \tag{36}$$

and

$$F' \cong \sum_{\gamma\gamma' \in \tilde{l}(G)} \sum_{\eta,\eta' \in \tilde{l}(H)} \mu(\gamma\gamma', \eta\eta') R^{H:H^{G,f}} \tag{37}$$

in accordance with the decomposition

$$Q^{\times N}/H = \cup_{\eta \in ekF} S(F, \eta) = \cup_{\eta \in ekF, \gamma \in ekA} T(a\gamma, \eta) \tag{38}$$

and

$$Q^{\times N}/G = \cup_{\gamma \in ekA} S(A, \gamma) = \cup_{\gamma \in ekA, \eta \in ekF} T(F\eta, \gamma) \tag{39}$$

respectively, into strata S and regions T in appropriate orbit sets. It is assumed in equations (34)–(37) that $G^f \in \gamma$, $H^f \in \eta$, $G^{H,f} \in \gamma'$, $H^{G,f} \in \eta'$.

Moreover,

$$T(F\eta, \gamma) = \cup_{\gamma',\eta'} \bar{D}(\gamma\gamma', \eta\eta')/G \tag{40}$$

and

$$T(A\gamma, \eta) = \cup_{\gamma',\eta'} \bar{D}(\gamma\gamma', \eta\eta')/H. \tag{41}$$

5. An example

We consider as an example the set $Q^{\times N}$ with Q being the set of vertices of a square (or, for example, as the star of a two-dimensional wavevector $\boldsymbol{k}$ with the C_{4v} symmetry), and $N = 4$. We put $Q = \{q = 1, 2, 3, 4\}$, and thus

$$Q^{\times 4} = \{(q_1, q_2, q_3, q_4)|q_j \in Q,\ j \in \tilde{N}\},\ |Q^{\times 4}| = 4^4 = 256. \tag{42}$$

We assume that G is the Pauli group $\sum_4$, and $H = C_{4v}$, the geometric point symmetry group of the square. The action of C_{4v} on Q is determined by permutations

$$C_4 = \begin{pmatrix} 1234 \\ 2341 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 1234 \\ 32\,14 \end{pmatrix} \tag{43}$$

in accordance with figure 1. This action constitutes the transitive representation $R^{C_{4v}:S_2^3}$, where S_2^3 is the stabilizer of the vertex $q = 1$.

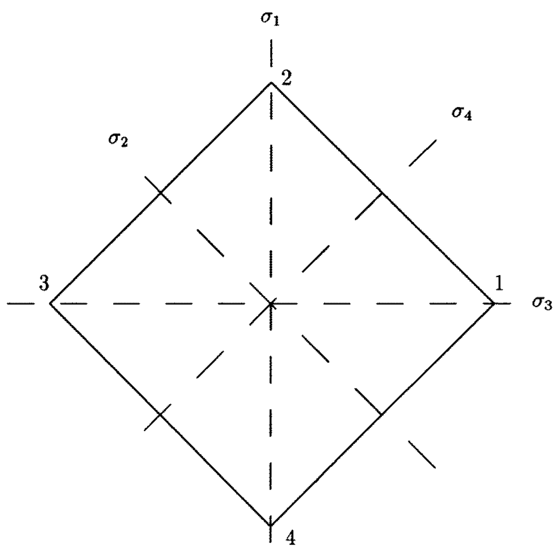

Figure 1. Labelling of vertices of the square and of vertical planes of the group C_{4v}.

The action A, as defined by equation (3), decomposes into transitive representations of the Pauli group $\sum_4$ as

$$A \cong 4R^{[4]} + 12R^{[31]} + 6R^{[2^2]} + 12R^{[21^2]} + R^{[1^4]} \tag{44}$$

where we use the abbreviation $R^{\mu} = R^{\Sigma_4:\Sigma^{\mu}}$ and $\Sigma^{\mu} = \Sigma_{\mu 1} \times \Sigma_{\mu 2} \times \Sigma_{\mu 3} \times \Sigma_{\mu 4} \subset \Sigma_4$ is the representative Young subgroup, defined by the partition μ of $N = 4$ ($\mu_1 + \mu_2 + \mu_3 + \mu_4 = 4$, $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \mu_4 \geqslant 0$, zeros omitted). Each term in equation (44) corresponds to a stratum $S(A, \mu)$ of $\sum_4$ on $Q^{\times 4}$, with epikernels labelled by partitions μ. The partition μ is simply the occupation of single-particle states within an N-particle orbit. The number of elements in the orbit is

$$|O^{\mu}_{\sum_4}| = \frac{N!}{\prod_{q \in Q} \mu_q!} \tag{45}$$

and the number of orbits in the stratum $[\mu]$ (integers in the r.h.s. of (44)) is

$$|S(A,\mu)| = \frac{n!}{\nu_0! \prod_{l\in\tilde{N}} \nu_l!} \tag{46}$$

where ν_l is the number of parts of μ of the length $l \in \tilde{N}$, and

$$\nu_0 = n - \sum_{l\in\tilde{N}} \nu_l \tag{47}$$

is the number of unoccupied single-particle states. Thus the dimensional check of equation (44) reads

$$4\cdot 1 + 12\cdot 4 + 6\cdot 6 + 12\cdot 12 + 1\cdot 24 = 4^4. \tag{48}$$

Dually, F is the geometric action of the group $H = C_{4v}$ in the fourth Cartesian power of Q, so that

$$F = \left(R^{C_{4v}:C_2^3}\right)^4 \cong 8R^{S_2} + 28R^{C_1} \tag{49}$$

where we again use an abbreviated notation $R^{C_{4v}:S_2^3} = R^{S_2}$, $R^{C_{4v}:C_1} = R^{C_1}$. Thus the action F exhibits two strata, with the epikernels S_2 and C_1. The dimensional check of equation (49) reads

$$\left(\frac{8}{2}\right)^4 = 8\cdot\frac{8}{2} + 28\cdot\frac{8}{1} \tag{50}$$

where the first factor in each term in the right-hand side is the number of orbits in the stratum, and the second is the number of elements in the orbit (the latter is presented as the fraction $|H|/|H^f|$ of the order of the group H to the order of appropriate stabilizer H^f).

Stratifications (44) and (49) yield only a global inventory of symmetric patterns in the set $Q^{\times 4}$. More detailed information is provided by considering each union of all orbits within a stratum as a set invariant under the dual action, and by separating all regions $\bar{T}(\gamma,\eta)$ and blocks b. This information is collected in table 1. It follows that there are seven regions, six of them consist of a single block, and one, $\bar{T}([21^2], C_1)$, encloses two blocks with distinct structures. Coarsening of actions A and F on various blocks, characterized by the degree $c(\gamma\gamma', \eta\eta')$, is inhomogeneous, and takes on the values 1 (four blocks), 2 (three blocks) and 8 (one block).

The set

$$ekA = \{[4], [31], [2^2], [21^2], [1^4]\} \tag{51}$$

of epikernels of the action of the Pauli group $\sum_4$ consists of all dominant partitions of the integer $N = 4$. Each epikernel $\gamma \in ekA$ corresponds to a definite distribution of particles over points of the single-particle configuration space Q.

The set

$$ekF = \{S_2, C_1\} \tag{52}$$

just results from all possible intersections of the single-particle stabilizers (S_2^1 or S_2^3, of figure 2) in a given four-particle configuration. Clearly, when N is increasing, then most of the orbits in the stratification corresponding to equation (5) belong to the trivial epikernel C_1, whereas the symmetric epikernel S_2 becomes more and more exceptional. Anyway, any increase of symmetry, i.e. any epikernel from the upper region of figure 3, is strictly forbidden.

Table 1. Block structure of the set $Q^{\times 4}$, imposed by the combinatoric version of the duality of Weyl, with $G = \sum_4$, $H = C_{4v}$. Symbol $D_{2d} \in \tilde{l}(\sum_4)$ in the column γ' denotes the conjugacy class of subgroups of $\sum_4$, which are isomorphic with the point group D_{2d} under the isomorphism $\sum_4 \sim T_d$.

Region		Block			
γ	η	γ'	η'	$\|b\|$	$c(\gamma\gamma', \eta\eta')$
[4]	S_2	[4]	S_2	4	1
[31]	S_2	[31]	S_2	16	1
	C_1	[31]	C_1	32	1
$[2^2]$	S_2	D_{2d}	C_{2v}	12	2
	C_1	D_{2d}	S_2'	24	2
$[21^2]$	C_1	$[21^2]$	C_1	96	1
	C_1	$[2^2]$	S_2	48	2
$[1^4]$	C_1	D_{2d}	C_{4v}	24	8
				$256 = 4^4$	

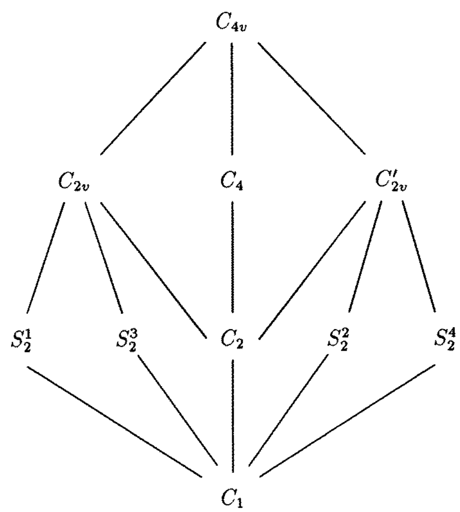

Figure 2. The lattice $\tilde{L}(C_{4v})$ of subgroups of the point group C_{4v}. Here $S_2^i = \{E, \sigma_i\}$ (cf figure 1). Lines indicate the partial order, which increases from the bottom (the unique minimal element C_1) to the top (the group C_{4v} itself).

An epikernel of F is insensitive to symmetric distribution of particles over the configuration space Q, since in this framework the particles are labelled by the elements of the set $\tilde{N}$ (equation (2)), and are thus distinguishable. The situation changes for the coarsened action F'. Now the group H of geometric symmetry of Q acts on orbits of the Pauli group, and thus on sets of unlabelled, i.e. indistinguishable particles, distributed over Q. Just the symmetry of this distribution is reflected in epikernels $\eta' \in ekF'$. We observe here an increase of symmetry, e.g. the region $\bar{T}([2^2], S_2)$ exhibits the symmetry C_{2v}, the non-symmetric region $\bar{T}([2^2], C_1)$ achieves the symmetry S_2, which does not exist in ekF, and the region $\bar{T}[[1^4], C_1]$ achieves the full geometric symmetry of the group C_{4v}. The latter result can be easily understood when we observe that the stratum $[1^4]$ of the Pauli group $\sum_4$ corresponds to the occupation of each point of Q by a single particle, so that

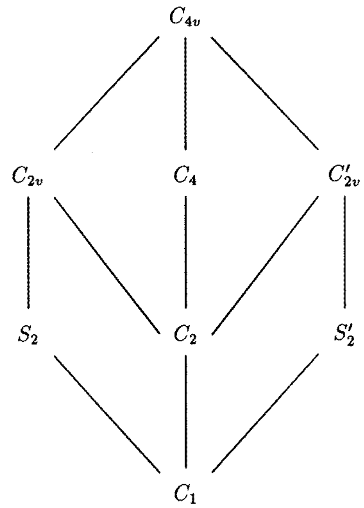

Figure 3. The lattice $\tilde{l}(C_{4v})$ of classes of conjugated subgroups.

all points are equally occupied. Clearly, it corresponds to a full geometric symmetry of the system of $N = 4$ particles.

The coarsened action F' in this case reads

$$F' \cong 2R^{C_1} + 3R^{S_2} + R^{S'_2} + R^{C_{2v}} + R^{C_{4v}} \tag{53}$$

with the dimensional check

$$2 \cdot 8 + 3 \cdot 4 + 4 + 2 + 1 = 35 = \binom{N+n-1}{N} \tag{54}$$

where the Newton symbol denotes the total number of orbits of the Pauli group $\sum_N$ on $Q^{\times N}$, with $N = n = 4$. Similarly, for the action A' we have

$$A' = R^{[4]} + 2R^{[31]} + R^{[21^2]} + R^{[2^2]} + 3R^{\sum_4 : D_{2d}} \tag{55}$$

with the demensional check

$$1 + 2 \cdot 4 + 12 + 6 + 3 \cdot 3 = 36. \tag{56}$$

The symbol D_{2d} in equation (55) denotes a subgroup in $\sum_4$, isomorphic with the point group D_{2d} under the isomorphism $\sum_4 \sim T_d$. It is worth observing that it is not any Young subgroup of $\sum_4$, so the coarsening of this action also yields an extra epikernel.

6. Final remarks and conclusions

We have proposed a finite analogy of the Weyl's duality between the symmetric and unitary groups acting on the Nth tensor power of an n-dimensional single-particle space. In our proposal, the single-particle space L of quantum states has been replaced by a finite set Q, equipped with the interpretation of a configuration space of a single particle on a lattice. The unitary group $U(n)$ is replaced by a subgroup $H \subset \sum_n$, $n = |Q|$, e.g. the geometric symmetry group of Q, and the Nth tensor power space $L^{\otimes N}$ by the Nth Cartesian power $Q^{\times N}$. The two mutually commuting actions, $A : G \times Q^{\times N} \to Q^{\times N}, G \subseteq \sum_N$, and $F : H \times Q^{\times N} \to Q^{\times N}, H \subseteq \Sigma_n$, impose on the set $Q^{\times N}$ the structure of strata, regions and blocks, in a far-reaching analogy with the original formulation of the duality of Weyl. In particular, we obtain a complete classification of all N-particle configurations.

Apparently, the structure of actions A and F is rather trivial: G acts on the set $Q^{\times N}$ from the right, and H from the left (cf equations (3) and (4)). As the result, the set $Q^{\times N}$ decomposes into blooks b, i.e. orbits of the direct product $H \times G$, each block characterized by a pair of epikernels (γ, η), $\gamma \in ek\,A$, $\eta \in ek\,F$. However, restrictions $A|_b$ and $F|_b$ of these actions to a block b are not completely complementary, i.e. the block b cannot, in general, be arranged into a rectangular matrix such that each row and column constitutes an orbit of G and H, respectively. We have described this situation, in a spirit of the duality of Weyl, by introducing new, coarsened actions of both groups on the set of orbits of the dual group, i.e. $A' : G \times Q^{\times N}/H$ and $F' : H \times Q^{\times N}/G$. These actions demonstrate transparently that the structure of a block $b \subset Q^{\times N}$ is determined not only by stabilizers G^f and H^f (or, more precisely, by the pair (γ, η) of epikernels), but also by the nature of mappings belonging to this block. The latter feature is implemented in the degree $c(\gamma\gamma', \eta\eta')$ of coarsening of the block b, given by equation (25). Richness of orbit structures on the set $Q^{\times N}$ arises from the observation that blocks with the same pair (γ, η) of epikernels can still exhibit different degrees of coarsening, and thus coarsenings of the whole set $Q^{\times N}$, related to each of the dual actions, are inhomogeneous.

The combinatorial variant of the duality of Weyl presented here is applicable for investigation of the configuration space of a system of N particles. In particular, it accounts for the indistinguishability of particles and geometric symmetry of the system *at the classical level.* We like to stress here that such configuration spaces have no natural linear structure, and thus our version of the Weyl duality has no immediate relations to state labelling problems of quantum systems. Such problems would arise after appropriate quantization of the system, say, within the Schrödinger picture, applied to an appropriate covering space of the classical configuration space. These problems exceed the scope of the present paper. The lack of any natural linear structure results here in replacement of the notion of an irreducible representation and an irreducible carrier space by a transitive representation and an orbit.

We argue that our approach is a version of the duality of Weyl, even if it is not placed in the realm of linear spaces. The feature which is shared with the original formulation of Weyl (1950), as well as with extensions of Moshinsky and Quesne (1970, 1971), consists in the observation that the repetition labels of the action of, say, the group G (i.e. labels of different carrier spaces of the same irreducible representation Γ of G in the linear case, or labels of orbits of the group G in our combinatorial approach) constitute a structure invariant under the dual action of the group H (an invariant subspace in the linear framework, or an invariant set in our case).

We have pointed out an interesting feature of the coarsened action F' in the case when H is the geometric symmetry group of the single-particle space Q. Then the action F does not reproduce the geometric symmetry at the level of N particles, mainly for the reason that they are treated as distinguishable entities. But the full recovery of geometric symmetry is done by coarsening of the action F, that is by the action F' on the set $Q^{\times N}/\sum_N$ of orbits of the Pauli group. This action thus sees only sets of indistinguishable particles, distributed over the elements of Q, and classifies these sets according to geometric symmetry, discarding the labels of particles. In this way we obtain in a natural manner some symmetries which are not represented by single-particle epikernels, but follow from geometric distribution of indistinguishable particles.

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